

Extraordinary waves in two dimensional electron gas with separate spin evolution and Coulomb exchange interaction

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Hydrodynamics analysis of waves in two-dimensional degenerate electron gas with the account of separate spin evolution is presented. The transverse electric field is included along with the longitudinal electric field. The Coulomb exchange interaction is included in the analysis. In contrast with the three-dimensional plasma-like mediums the contribution of the transverse electric field is small. We show the decrease of frequency of both the extraordinary (Langmuir) wave and the spin-electron acoustic wave due to the exchange interaction. Moreover, spin-electron acoustic wave has negative dispersion at the relatively large spin-polarization. Corresponding dispersion dependencies are presented and analyzed.

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I. INTRODUCTION

Collective excitations in spin-polarized electron gas are considered as waves propagating in a plasma-like medium. The quantum hydrodynamic (QHD) method [1], [2] and its generalization for the separate spin evolution (SSE) [3] are applied for the theoretical modeling of this system.

The SSE-QHD method is developed for the spin-polarized degenerate electron gas [3]. Majority of papers on this subject are focused on three-dimensional electron gas existing in magnetically ordered metals (see for instance [4], [5]). However, it can be applied to the spin-polarized electron-hole liquid in the regime of degenerate carriers. It can be done similarly to the analysis of the electron-positron plasmas [6]. Including difference of the effective masses of the electrons and holes or (and) different concentrations of the electrons and holes, we find more similarity between electron-hole liquid and electron-positron-ion plasmas. Approximately, the SSE-QHD may be applied for non-degenerate electron-hole liquid, but it requires a modified equation of state and account of the scattering processes leading to the damping of excitations.

The SSE-QHD allows to consider linear [4], [6] and nonlinear [7] bulk spin-electron acoustic waves (SEAWs). It also allows to consider the surface SEAWs [5] and SEAWs in two-dimensional electron gas [8].

Spectrum of the plasma-like mediums is highly affected by the transverse electric field at the wave propagation perpendicular to the external field. The transverse electric field leads to formation of the extraordinary waves instead of the Langmuir or upper hybrid waves for the electrostatic regime [9]. Same effect was demonstrated for the SEAWs [10]. Therefore, we pay special attention

to the transverse electric field in two-dimensional electron gas.

The collective excitations in the two-dimensional electron gas with no account of the spin polarization have been studied for a long time [11], [12], including the magnetoplasmons existing in the two-dimensional electron gas located in the external magnetic field [13], [14]. The spin-polarized two-dimensional electron gas exists in the magnetic semiconductors. The study of SEAWs in two-dimensional electron gas appeared by analogy with bulk properties of the three-dimensional electron gas [3], [4]. However, there is a relatively long history of theoretical and experimental study of the spin plasmons in the spin-polarized two-dimensional electron gas (see for instance [15], [16]). The spin plasmons show dispersion and physical mechanism to the SEAWs. This similarity shows that described above linear and nonlinear, bulk and surface SEAWs can be considered as spin plasmons in three-dimensional electron gas.

It seems that all analysis of spin plasmons [15] is focused on the quantum wells in the magnetized semiconductors. Acoustic plasmon frequency is experimentally measured and the results are compared with the values calculated for spin polarized electrons and holes in p-type A_3B_5 semiconductors [16]. Spin flip waves and spin density fluctuations of a two-dimensional spin-polarized electron-gas in a semimagnetic $Cd_{10.008}Mn_{0.008}Te$ quantum well are considered in Ref. [17]. A maximum value of the spin polarization degree of 35 percent is deduced from measurements. Collective excitations in the spin polarized quantum well were also considered in Refs. [18], [19], [20], [21]. Recently, the contribution of Rashba spin-orbit coupling in the collective modes in two- and three-dimensional electron systems considered in Refs. [22], [23].

Large contribution of the exchange interaction in the two-dimensional electron gas of GaAs microstructures is demonstrated experimentally [24] and theoretically within the local density approximation [25]. Hence, we

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consider the Coulomb exchange interaction contribution in spectrum of the Langmuir waves (plasmons) and the SEAWs (spin plasmons) in terms of the SSE-QHD. The influence of the exchange interaction on the collective effects in a quantum well was analyzed in Ref. [26].

II. MODEL

We apply the SSE-QHD model developed in Ref. [3], [4] and adopted for two-dimensional systems in Ref. [8]. Since we are interested in analysis of the effects of Coulomb exchange interaction we use generalization of the SSE-QHD containing the exchange interaction [7]. However, Ref. [7] contains the exchange interaction for the three-dimensional electron gas. We need to substitute it in the similar way by the exchange interaction in two-dimensional electron gas in accordance with Refs. [27], [28]. We also use the full integral Maxwell equations for the expressions of the electric and magnetic fields to consider the contribution of the transverse electric field in the waves in magnetized spin-polarized electron gas. Overall we have the following set of four hydrodynamic equations: the continuity equations

$$\partial_t n_s + \nabla(n_s \mathbf{v}_s) = \pm \frac{\gamma}{\hbar} (S_x B_y - S_y B_x), \quad (1)$$

and the Euler equations

$$\begin{aligned} m n_s (\partial_t + \mathbf{v}_s \nabla) \mathbf{v}_s + \nabla p_s \\ = q_e n_s \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_s, \mathbf{B}] \right) + \mathbf{F}_{SS,s}, \end{aligned} \quad (2)$$

with the force field of spin-spin interaction

$$\begin{aligned} \mathbf{F}_{SS,s} = \pm \gamma_e n_s \nabla B_z + \frac{\gamma_e}{2} (S_x \nabla B_x + S_y \nabla B_y) \\ \pm \frac{m \gamma_e}{\hbar} [(\mathbf{J}_{(M)x} - \mathbf{v}_s S_x) B_y - (\mathbf{J}_{(M)y} - \mathbf{v}_s S_y) B_x], \end{aligned} \quad (3)$$

containing the spin current

$$\mathbf{J}_{(M)\alpha} = \frac{1}{2} (\mathbf{v}_u + \mathbf{v}_d) S_\alpha - \varepsilon_{\alpha\beta z} \frac{\hbar}{4m} \left(\frac{\nabla n_u}{n_u} - \frac{\nabla n_d}{n_d} \right) S_\beta, \quad (4)$$

and the effective pressure

$$p_s = \pi \hbar^2 n_s^2 / m - \zeta \frac{8\beta}{3\pi \sqrt{\pi}} q_e^2 n_d^{\frac{3}{2}} \delta_{sd}, \quad (5)$$

where we introduce $\beta \equiv 24 \text{arsh} 1 = 24 \ln(1 + \sqrt{2}) = 21.153$ and

$$\zeta = 1 - \frac{(1 - \eta)^{3/2}}{(1 + \eta)^{3/2}}. \quad (6)$$

The electromagnetic field is caused by the charges, electric currents and magnetic moments (spin) $\mathbf{E} = -\nabla \varphi - \partial_t \mathbf{A}/c$, $\mathbf{B} = \mathbf{B}_{ext} + \nabla \times \mathbf{A}_j + \mathbf{B}_{spin}$, where

$$\varphi(\mathbf{r}, t) = \sum_{s=u,d} q_e \int \frac{n_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (7)$$

$$\mathbf{A}_j(\mathbf{r}, t) = \sum_{s=u,d} \frac{q_e}{c} \int \frac{\mathbf{j}_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (8)$$

and

$$\mathbf{B}_{spin}(\mathbf{r}, t) = \int [(\mathbf{M}(\mathbf{r}', t') \nabla) \nabla - \mathbf{M}(\mathbf{r}', t') \Delta] \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (9)$$

where $\mathbf{M}(\mathbf{r}', t') = \mathbf{M}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)$ is the magnetization existed in point with coordinate \mathbf{r}' in an earlier moment of time t' , with $d\mathbf{r}' = dx' dy'$ is the element of volume in 2D space.

III. LINEARIZED EQUATIONS

We consider the propagation of plane waves along the Ox direction: $\mathbf{k} = \{k_x, 0, 0\}$. In the longitudinal waves (the electrostatic waves), the perturbation of electric field is parallel to the direction of wave propagation. However, we consider the longitudinally-transverse waves. Hence, we include the electric field perturbation along the Oy direction: $\mathbf{E} = \{E_x, E_y, 0\}$.

$$-i\omega \delta n_s + i k_x n_{0s} \delta v_{sx} = 0, \quad (10)$$

$$-i\omega m n_{0s} \delta \mathbf{v}_s + i \mathbf{k} \delta p_s = q_e n_{0s} \delta \mathbf{E} + m n_{0s} \Omega_e [\delta \mathbf{v}_s, \mathbf{e}_z], \quad (11)$$

where $\Omega_e = q_e B_0 / mc$ is the cyclotron frequency. We do not include the spin-spin interaction force field in the Euler equation as a small effect.

The perturbation of electric field has the following connection with the perturbation of concentration and the velocity field

$$\begin{aligned} \delta \mathbf{E} = -q_e \sum_{s=u,d} \left(\nabla \int \frac{\delta n_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \right. \\ \left. + \frac{n_{0s}}{c^2} \partial_t \int \frac{\delta \mathbf{v}_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \right). \end{aligned} \quad (12)$$

As it is demonstrated in Appendix for plane waves, the perturbation of the electric field can be presented in the following form:

$$\delta \mathbf{E} = i q_e \sum_{s=u,d} \left(-\Im \mathbf{k} \delta n_s + \Im \frac{\omega}{c^2} n_{0s} \delta \mathbf{v}_s \right). \quad (13)$$

As we will see below the result of application of equations (10) and (11) is rather complicate. Thus, as an intermediate step we consider the extraordinary waves in 2DEG in terms of the single fluid model of electrons.

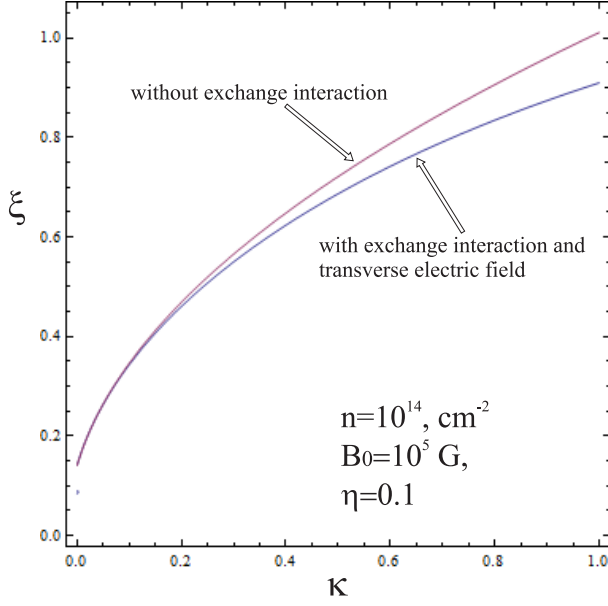


FIG. 1: (Color online) The figure shows the dimensionless dispersion dependence ω/ω_0 on the dimensionless wave vector $\kappa = k/\sqrt{n_0}$, where $\omega_0^2 = 2\pi e^2 n_{0e}^{3/2}/m$ is a constant giving a characteristic frequency. Parameters are presented in the figure.

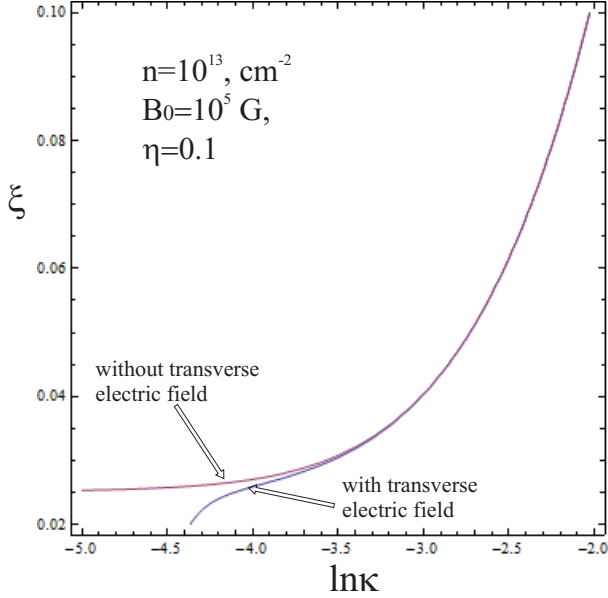


FIG. 2: (Color online) The figure shows a considerable change of the dispersion dependence at the small wave vectors due to the transverse electric field of wave. We present the dimensionless frequency ξ as a function of the natural logarithm of the dimensionless wave vector $\ln \kappa$.

IV. SINGLE FLUID MODEL OF ELECTRONS AND EXTRAORDINARY WAVE DISPERSION

Linearized set of quantum hydrodynamic equations for electrons considered as a single fluid is:

$$-\omega \delta n_e + \imath k_x n_{0e} \delta v_{ex} = 0, \quad (14)$$

$$-\omega n_{0e} \delta v_{ex} + \imath k_x U_e^2 \delta n_e = \frac{q_e}{m} n_{0e} \delta E_x + n_{0e} \Omega_e \delta v_{ey}, \quad (15)$$

$$-\omega n_{0e} \delta v_{ey} = \frac{q_e}{m} n_{0e} \delta E_y - n_{0e} \Omega_e v_{ex}, \quad (16)$$

where

$$\delta E_x = q_e \Im \left(-k_x \delta n_e + \frac{\omega}{c^2} n_{0e} \delta v_x \right), \quad (17)$$

$$\delta E_y = q_e \Im \frac{\omega}{c^2} n_{0e} \delta v_y, \quad (18)$$

and $U_e^2 = (1 + \eta^2) \frac{\pi \hbar^2 n_{0e}}{m_e^2} - (1 + \eta)^{3/2} \zeta \frac{\beta \sqrt{2\pi} e^2}{\pi^2 m_e} \sqrt{n_{0e}}$.

Equations (14)-(18) lead to the following dispersion equation for the longitudinally-transverse waves in spin-polarized two-dimensional electron gas

$$\omega^2 \left(1 + \frac{1}{\sqrt{1 - \frac{\omega^2}{k^2 c^2}}} \frac{\omega_{Le}^2}{k^2 c^2} \right) - k^2 U_e^2 - \frac{1}{\sqrt{1 - \frac{\omega^2}{k^2 c^2}}} \omega_{Le}^2 = \frac{\Omega_e^2}{1 + \frac{1}{\sqrt{1 - \frac{\omega^2}{k^2 c^2}}} \frac{\omega_{Le}^2}{k^2 c^2}}, \quad (19)$$

where we have used the two dimensional Langmuir frequency

$$\omega_{Le}^2 = \frac{2\pi e^2 k n_{0e}}{m} \sim k. \quad (20)$$

To drop the transverse electric field contribution we should consider the limit case $c \rightarrow \infty$. Thus, we find the dispersion dependence for the longitudinal waves in 2DEG in the single fluid model of electrons (see for instance [28])

$$\omega^2 = \Omega_e^2 + \omega_{Le}^2 + U_e^2 k^2. \quad (21)$$

In both regimes we find single branch of the dispersion dependence.

Considering the linear perturbations we can compare the contributions of the Fermi pressure and the exchange interaction. Their relative behavior can be described by the following dimensionless parameter: $\Lambda = \frac{\hbar^2}{m_e^2} \sqrt{n_{0e}}$ [28]. More precisely, the Fermi pressure is larger than the Coulomb exchange interaction if the concentrations of electrons satisfy the following condition

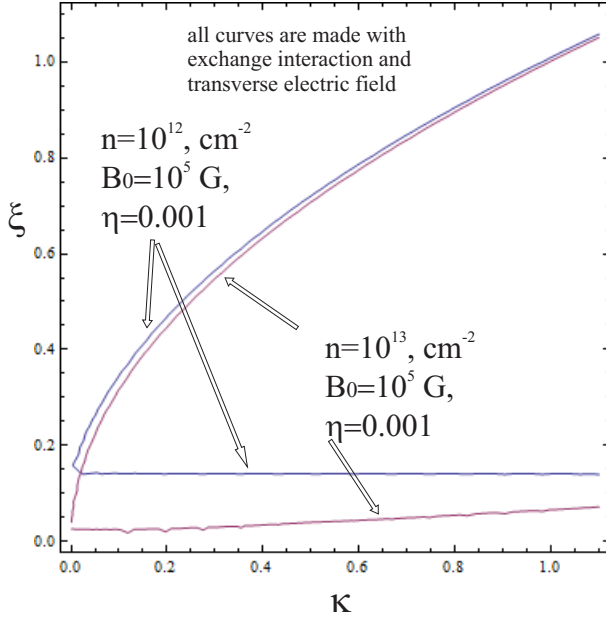


FIG. 3: (Color online) The figure shows solutions of dispersion equation (25) for regimes of two different concentrations at the fixed spin polarization. In each regime we find two solutions: the Langmuir wave and the SEAW.

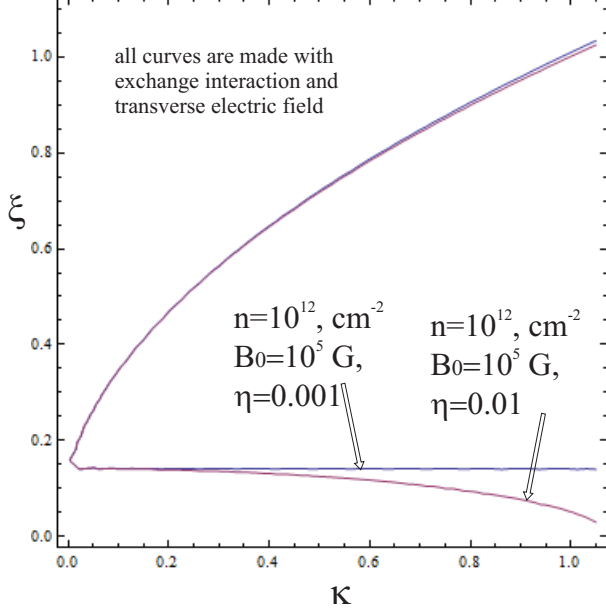


FIG. 4: (Color online) The figure shows dispersion dependencies arising as solutions of equation (25) for regimes of two different spin polarizations at the same concentrations and external magnetic field. We assume that inner interaction with ions of the medium give dominant contribution to the spin polarization.

$\sqrt{n_{0d}} > \frac{2\zeta\sqrt{\pi\beta}}{\pi^3(1+\eta^2)} \frac{me^2}{\hbar^2}$. It gives $n_{0d}(\eta = 0.1) = 6 \times 10^{16} \text{ cm}^{-2}$ or $n_{0d}(\eta = 0.01) = 7 \times 10^{15} \text{ cm}^{-2}$.

Numerical analysis shows that the change of the dispersion dependence due to the transverse electric field is very small, it arises in the regime of small wave vectors only. Formulae (21) and (19) gives coinciding curves in Fig. 1. The small change of spectrum exists at the small wave vectors, as it is demonstrated in Fig. 2. Comparing two curves in Fig. 1 we see that the Coulomb exchange interaction gives considerable decrease of the dispersion dependence of the Langmuir waves or the extraordinary waves with the small contribution of the transverse electric field.

V. DISPERSION DEPENDENCE FOR TWO-FLUID MODEL OF ELECTRONS

After the calculation of electromagnetic potentials the linear SSE hydrodynamic equations (10), (11) can be presented in a local form as a set of algebraic equations:

$$\delta n_s = n_{0s} k_x \delta v_{xs} / \omega, \quad (22)$$

$$-\omega m n_{0s} \delta v_{sx} + i k_x m U_s^2 \delta n_s = q_e^2 n_{0s} i \Im \left(-k_x \delta n_u - k_x \delta n_d \right. \\ \left. + \frac{\omega}{c^2} n_{0u} \delta v_{ux} + \frac{\omega}{c^2} n_{0d} \delta v_{dx} \right) + m n_{0s} \Omega_e \delta v_{sy}, \quad (23)$$

$$-\omega m n_{0s} \delta v_{sy} = q_e^2 n_{0s} i \Im \left(\frac{\omega}{c^2} n_{0u} \delta v_{uy} \right. \\ \left. + \frac{\omega}{c^2} n_{0d} \delta v_{dy} \right) - m n_{0s} \Omega_e \delta v_{sx}, \quad (24)$$

where $U_s^2 = \frac{2\pi\hbar^2}{m^2} n_{0s} - \zeta \frac{4\beta e^2}{\pi\sqrt{\pi} m_e} \sqrt{n_{0d}} \delta_{sd}$.

Substituting formula (22) to equations (23) and (24) we obtain a set of four uniform algebraic equations. This set has a nonzero solution if its determinant is equal to zero. It leads to the following dispersion equation:

$$\left(1 + \frac{\omega_{Ru}^2 + \omega_{Rd}^2}{k^2 c^2}\right) \left[(\omega^2 - k^2 U_u^2)(\omega^2 - k^2 U_d^2) - [\omega_{Ru}^2(\omega^2 - k^2 U_d^2) + \omega_{Rd}^2(\omega^2 - k^2 U_u^2)] \left(1 - \frac{\omega^2}{k^2 c^2}\right) \right] - \Omega_e^2 \left\{ \left[\omega^2 - k^2 U_u^2 - \omega_{Ru}^2 \left(1 - \frac{\omega^2}{k^2 c^2}\right) \right] \left(1 + \frac{\omega_{Ru}^2}{k^2 c^2}\right) + \left[\omega^2 - k^2 U_d^2 - \omega_{Rd}^2 \left(1 - \frac{\omega^2}{k^2 c^2}\right) \right] \left(1 + \frac{\omega_{Rd}^2}{k^2 c^2}\right) \right\} + \Omega_e^4 = 0, \quad (25)$$

where $\omega_{Rs}^2 = q_e^2 n_{0s} k^2 \Im / m = \omega_{Ls}^2 / \sqrt{1 - \omega^2 / k^2 c^2}$ is the retarding Langmuir frequency.

In the electrostatic limit ($c \rightarrow \infty$) we can drop the transverse electric field contribution in equation (25) and find the following dispersion equation:

$$(\omega^2 - \omega_{Lu}^2 - \Omega_e^2 - k^2 U_u^2)(\omega^2 - \omega_{Ld}^2 - \Omega_e^2 - k^2 U_d^2) - \omega_{Lu}^2 \omega_{Ld}^2 = 0. \quad (26)$$

It was derived in Ref. [8] (the dispersion equation is not presented in the paper, but its solution is presented by formula 7), it is also similar to results of earlier Ref. [15], starting from the SSE-QHD in the electrostatic regime.

Both equations (25) and (26) have two solutions describing two branches of the wave dispersion dependence: the Langmuir (plasmon) mode and the spin-electron acoustic mode (spin-plasmon mode). Equation (25) describes these waves in the regime of longitudinally-transverse waves. In this case, similarly to three-dimensional plasma-like mediums, we can call these wave the extraordinary waves.

At the account of the SSE we find numerically negligibly small contribution of the transverse electric field. Hence, equations (25) and (26) give coinciding curves in Figs. 3, 4.

In Fig. 3 we see that at larger concentrations the dimensionless frequency of the Langmuir waves (the pair of upper curves) has smaller value and growth rate. The SEAWs (the pair of lower curves) have considerable smaller dimensionless frequencies at the larger concentration, but the growth rate becomes larger at the larger concentration.

As we see from Fig. 4 the increase of the spin polarization leads to the decrease of the frequencies of both waves. This effect reveals itself more in the SEAW spectrum. The exchange interaction decreases the SEAW frequency down to negative group velocity. Further increase of the spin polarization decreases the area of SEAW existence since its frequency goes down to zero value at the accessible wave vectors $\kappa < 1$.

VI. CONCLUSION

We have considered waves in the magnetized two-dimensional electron gas located in the external constant uniform magnetic field directed perpendicular to the plane. We have paid attention to both the longitudinal and the transverse parts of the electric field of the

wave perturbation propagating in the spin-polarized degenerate two-dimensional electron gas. Considering all electrons as the single fluid we find one wave solution: the extraordinary wave or the hybrid wave in the electrostatic limit. It was recently demonstrated that the account of the SSE in 2DEG with equilibrium spin polarization leads to the second branch: the SEAW. It was done in the longitudinal (the electrostatic limit) regime. Change of the dispersion dependence of the SEAWs at the account of the transverse electric field contribution in the wave propagation which forms the extraordinary SEAW has been demonstrated.

The influence of the Coulomb exchange interaction on the propagation of waves in 2DEG has been studied in both regimes: the single fluid model of electrons and the SSE-QHD.

VII. ACKNOWLEDGEMENTS

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VIII. APPENDIX: INTEGRALS FOR POTENTIALS OF ELECTROMAGNETIC FIELD

$$\delta\varphi = q_e \sum_{s=u,d} \int \frac{\delta n_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad (27)$$

$$\delta\mathbf{A} = \frac{q_e}{c} \sum_{s=u,d} n_{0s} \int \frac{\delta \mathbf{v}_s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad (28)$$

We consider the plane wave perturbations $\delta f = F e^{-i\omega t + i\mathbf{k}\mathbf{r}}$ which leads to

$$\begin{aligned} & \int \frac{\delta f(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \\ & = F \int \frac{e^{(-i\omega(t - |\mathbf{r} - \mathbf{r}'|/c) + i\mathbf{k}\mathbf{r}')} }{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \end{aligned}$$

$$\begin{aligned}
&= F e^{-i\omega t + i\mathbf{k}\mathbf{r}} \int \frac{e^{(i\omega(|\mathbf{r}-\mathbf{r}'|/c) + i\mathbf{k}(\mathbf{r}'-\mathbf{r}))}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \\
&= \delta f \int e^{i\omega\xi/c + i k \xi \cos\varphi} d\varphi d\xi = \Im \delta f, \quad (29)
\end{aligned}$$

where

$$\Im \equiv \int e^{i\omega\xi/c + i k \xi \cos\varphi} d\varphi d\xi. \quad (30)$$

As the result of calculation of integral (30) we find

$$\Im = \frac{2\pi}{k} \frac{1}{\sqrt{1 - \frac{\omega^2}{k^2 c^2}}}. \quad (31)$$

Substituting our results in the expressions for $\delta\varphi$ and $\delta\mathbf{A}$ we have

$$\delta\varphi = q_e \Im (\delta n_u + \delta n_d), \quad (32)$$

$$\delta\mathbf{A} = \frac{q_e}{c} \Im (n_{0u} \delta \mathbf{v}_u + n_{0d} \delta \mathbf{v}_d). \quad (33)$$

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